RENORMING AM-SPACES

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Abstract. We prove that any separable AM-space X has an equivalent lattice norm for which no non-trivial surjective lattice isometries exist. Moreover, if X has no more than one atom, then this new norm may be an AM-norm. As our main tool, we introduce and investigate the class of so called Benyamini spaces, which "approximate" general AM-spaces.

1. Introduction

The question of renormings has been extensively studied in the Banach space literature. The goal is to equip a prescribed Banach space with an equivalent norm in a way that alters its isometric properties in a certain desirable way (the isomorphic properties meanwhile remain the same). Many results of this type appear in [4]; for more modern treatment see [5] or [6].

We are interested in producing a renorming with a prescribed group of isometries (throughout this paper, all isometries are assumed to be linear and surjective unless specified otherwise). One of the first results appeared in [2]; there, it was shown that any separable real Banach space can equipped with an equivalent norm for which there are only two isometries – the identity and its opposite. The separability assumption was later removed in [10]. More recent papers [7], [8], [9] deal with renorming a separable Banach space in a way that produces a prescribed group of isometries.

In this work, we consider lattice renormings of separable AM-spaces. Recall that an AM-space is a Banach lattice in which the equality $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for any positive x and y; a lattice norm with this property is called an AM-norm. We also restrict oursleves to lattice isometries – that is, surjective (linear) isometries which preserve lattice operations. Our main result is:

Theorem 1.1. *Suppose $(X, \|\cdot\|)$ is a separable AM-space, and $c > 1$. Then X can be equipped with an equivalent lattice norm $|||\cdot|||$ so that $\|\cdot\| \leq |||\cdot||| \leq c\|\cdot\|$, and the identity map is the only lattice isometry on $(X.|||\cdot|||)$. If X has no more than one atom, then $|||\cdot|||$ can be chosen to be an AM-norm.*

The restriction on the number of atoms is essential; see Remark 3.8.

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The proof of Theorem 1.1 proceeds in two steps. In Section 2, we introduce a new class of AM-spaces, which we call "Benyamini spaces," for their original discoverer [3]. We establish that any separable AM-space can be transformed into a Benyamini space with arbitrarily small distortion; this result may be interesting in its own right. In Section 3 we renorm a Benyamini space, eliminating all non-trivial isometries (the new norm may cease to be an AM-norm if more than one atom is present). The proof of Theorem 1.1 then follows by combining Proposition 2.3 and Theorem 3.1.

Throughout this paper, we use the standard functional analysis facts and notation. For more detail, the reader is referred to e.g. [11] and [12]. All spaces are assumed to be separable, and the field of scalars is that of real numbers. For a normed space Y, we use the notation $\mathbf{B}(Y) = \{y \in Y : \|y\| \leq 1\}$. If Y is an ordered space and $A \subset Y$, we denote by A_+ the positive part of A – that is, $\{a \in A : a \geq 0\}$.

2. Benyamini spaces

Here we investigate a class of AM-spaces – the Benyamini spaces. Such spaces are flexible: any separable AM-space can be transformed into a space of this form (Proposition 2.3). On the other hand, Benyamini spaces can be easily analyzed, since they have a concrete representation, similar to a $C(K)$ space. In particular, we describe atoms in, and duals of, such spaces in Subsections 2.3 and 2.4, respectively.

2.1. Definition and basic properties.

Definition 2.1. *We say that a Banach lattice X is a C-*Benyamini space *(the constant $C > 1$ will often be omitted) if it is a sublattice of $C(K)$, where:*

(1) K is the one point compactification of the union of mutually disjoint compact sets K_n ($K = (\cup_n K_n) \cup \{\infty\}$).
(2) $X \subset C_0(K)$ – that is, any $x \in X$ vanishes at ∞.
(3) X separates points for each K_n (that is, for all $t, s \in K_n$, there exists $x \in X$ such that $x(t) \neq x(s)$).
(4) If $t \in K_m$, $s \in K_n$, and for all $x \in X$, $x(t) = \lambda x(s)$ for some fixed λ, then $\lambda = C^{n-m}$.

Note that, if X is separable, then each K_n is metrizable (due to (3)). Consequently, K is metrizable.

We begin by establishing some properties of Benyamini spaces.

Lemma 2.2. *Let K, C, and $X \subseteq C_0(K)$ be as above. For $n \neq m$, define $D(m, n)$ by*

$$D(m, n) := \big\{t \in K_m : \exists s \in K_n \text{ such that } \forall x \in X, x(t) = C^{n-m} x(s)\big\}.$$

Then $D(n, m)$ is closed and homeomorphic to $D(m, n)$.

Proof. Define $\phi_{mn} : D(m,n) \to D(n,m)$ by setting $\phi_{mn}(t)$ to be the unique $s \in D(n,m)$ with the property that $x(t) = C^{n-m}x(s)$ for any $x \in X$. Because X separates points for each K_n, ϕ_{mn} is well defined and injective. By definition of $D(m,n)$, it is bijective, and $\phi_{nm} = \phi_{mn}^{-1}$. To show continuity, suppose $t_k \to t \in D(m,n)$ with $t_k \in D(m,n)$. Suppose there exists a subsequence $s_j = \phi_{mn}(t_{k_j}) \to s \neq \phi_{mn}(t)$ (we can limit ourselves to such a case since K_m is compact). Then for all $x \in X$,

$$x(s) = \lim x(s_j) = \lim_j C^{m-n}x(t_{k_j}) = C^{m-n}x(t) = x(\phi_{m,n}(t)),$$

which is a contradiction, since X separates points. To prove that $D(m,n)$ is closed, suppose $t_k \to t$. Then for all $x \in X$,

$$x(t) = \lim_k x(t_k) = C^{n-m} \lim_k x(\phi_{mn}(t_k)).$$

By compactness, we assume that $\phi_{mn}(t_k) \to s \in K_n$. Hence for all $x \in X, x(t) = C^{n-m}x(s)$, so $s \in D(m,n)$. □

The importance of Benyamini spaces stems from the fact that any separable AM-space can be "approximated" by a Benyamini space.

Proposition 2.3. *If X is a separable AM-space, then for every $C > 1$ there exists a Benyamini space X' and a surjective lattice isomorphism $\Phi : X \to X'$ so that for all $x \in X$, $\|x\| \leq \|\Phi(X)\| \leq C\|x\|$.*

The proof below is similar to that of [3, Lemma 1].

Proof. We can assume that $X \subset C(H)$ for some Hausdorff compact H. First, as in [3], we consider the set $F := \cap_{x \in X} x^{-1}(0)$. If $F \neq \emptyset$, identify F with a single point z by passing from K to K/F. Let x_n be a dense sequence in $\mathbf{B}(X)_+$. Let $\psi = (C-1)\sum_{n=1}^{\infty} C^{-n}x_n$. Clearly ψ belongs to X.

Let $H_n = \{t \in H : C^{-n} \leq \psi(t) \leq C^{-n+1}\}$. If infintely many H_n's are non-empty, let $\widetilde{H}_n$ be disjoint copies of H_n, and let $\widetilde{H} = (\cup_n \widetilde{H}_n) \bigcup \{\infty\}$ be the one point compactification of $\bigcup \widetilde{H}_n$. Otherwise, let $\widetilde{H} = \bigcup_n \widetilde{H_n}$. Define the map $\Psi : \widetilde{H} \to H$ sending $\widetilde{H}_n$ to H_n and ∞ to z. Note that if F is empty, then $\psi(t) > 0$ for all $t \in K$, and since ψ itself is continuous, its image is compact and so must be bounded below; then $H_n = \emptyset$ for n large enough. Otherwise, ψ vanishes only at z. In either case, Ψ is a continuous surjection from $\widetilde{H}$ onto H, which implies that $C(H)$ embeds into $C(\widetilde{H})$ isometrically via the map $x \mapsto \widetilde{\Psi}x := x \circ \Psi$.

Now define a lattice isomorphism $U : C_0(\widetilde{H}) \to C_0(\widetilde{H})$ by setting, for $x \in C_0(\widetilde{H})$, $[Ux](\infty) = 0$, and $[Ux](t) = \dfrac{C^{1-n}x(t)}{(\widetilde{\Psi}\psi)(t)}$. Observe that $\|Ux\| \leq \|x\| \leq C\|x\|$. Then $T = U \circ \widetilde{\Psi}$ is a lattice homomorphism, and $Y = T(X)$ is a sublattice of $C_0(\widetilde{H})$. We claim that, if $t \in \widetilde{H}_m$ and $s \in \widetilde{H}_n$ are such that $y(t) = \lambda y(s)$ for any $y \in Y'$, then $\lambda = C^{n-m}$. Indeed, $y = Tx$ for some $x \in X$, so

$$\lambda = \frac{y(t)}{y(s)} = \frac{C^{1-m}x(\Psi(t))}{\psi(\Psi(t))} \cdot \frac{\psi(\Psi(s))}{C^{1-n}x(\Psi(s))} = C^{n-m} \cdot \frac{x(\Psi(t))}{x(\Psi(s))} \cdot \frac{\psi(\Psi(s))}{\psi(\Psi(t))}.$$

From this, it follows that $x(\Psi(t))/x(\Psi(s))$ is a constant on X. Either $\Psi(t) = \Psi(s)$, or $t' = \Psi(t)$ and $s' = \Psi(s)$ are "defining points" for $X \subset C(H)$ – that is, $x(t')/x(s')$ is independent of $x \in X$. Either way, $\lambda = C^{n-m}$.

Finally, we transform the sets $\widetilde{H}_n$ into sets K_n, whose points are separated by X'. By the preceding paragraph, if $t, s \in \widetilde{H}_n$ are such that $y(t) = \lambda y(s)$ for any $y \in Y$, then $\lambda = 1$. Define an equivalence relation on $\widetilde{H}$: $t \sim s$ if for all $y \in Y$, $y(s) = y(t)$. Clearly the equivalence classes are closed, hence each quotient space $K_n := \widetilde{H}_n/\sim$ is compact. Identify $\widetilde{H}/\sim$ with $K = (\cup_n K_n) \cup \{\infty\}$, which is the one-point compactification of $\cup_n K_n$. Define $\Phi : Y \to C_0(K)$ by setting, for $y \in Y$, $[\Phi y]([t]) = y(t)$, where $[t]$ is the equivalence class of t. Clearly Φ is a lattice isometry. $X' = \Phi(Y)$ is a Benyamini space, and $\Phi \circ T : X \to X'$ is a lattice isomorphism with desired properties. □

Remark 2.4. The Benyamini space X', constructed from X using Proposition 2.3, may have a different group of isometries. We do not know whether the Benyamini space can be constructed while preserving the group of isometries (or even a subgroup thereof).

2.2. **Extension of functions in Benyamini spaces.** We say that a function $x \in C(K_M \cup \ldots \cup K_N)$ is *consistent* if $x(s) = C^{n-m}x(\phi_{mn}(s))$ whenever $s \in D(m, n)$, with $M \leq n, m \leq N$. We shall say that a family of functions $x_n \in C(K_n)$ ($M \leq n \leq N$) is *consistent* if the function $x \in C(K_M \cup \ldots \cup K_N)$, defined via $x|_{K_n} = x_n$, is consistent.

Proposition 2.5. *(1) If $L \leq N$, and $x \in C(K_L \cup \ldots \cup K_N)$ is a consistent function, then there exists $\tilde{x} \in X$ so that $\tilde{x}|_{K_1 \cup \ldots \cup K_N} = x$, and, for $j \notin \{L, \ldots, N\}$, $\sup_{K_j} |\tilde{x}| \leq \max_{L \leq i \leq N} C^{i-j} \sup_{K_i} |x|$.*

(2) If, furthermore, $y \in X_+$ is such that $0 \leq x \leq y$ on $K_L \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that, in addition, $0 \leq \tilde{x} \leq y$.

Remark 2.6. In a similar fashion, one can show that if $y, z \in X$ are such that $z \leq x \leq y$ on $K_M \cup \ldots \cup K_N$, then $\tilde{x}$ can also be selected in such a way that $z \leq \tilde{x} \leq y$.

The proof of Proposition 2.5 is obtained by combining Lemmas 2.7 and 2.9. First we deal with "downward" extensions.

Lemma 2.7. *(1) If $x \in C(K_1 \cup \ldots \cup K_N)$ is a consistent function, then there exists $\tilde{x} \in X$ so that $\tilde{x}|_{K_1 \cup \ldots \cup K_N} = x$, and, for $j > N$, $\sup_{K_j} |\tilde{x}| \leq \max_{1 \leq i \leq N} C^{i-j} \sup_{K_i} |x|$.*

(2) If, furthermore, $y \in X_+$ is such that $0 \leq x \leq y$ on $K_1 \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that, in addition, $0 \leq \tilde{x} \leq y$.

Proof. (1) We define $\tilde{x}$ recursively. Suppose $\tilde{x}|_{K_1 \cup \ldots \cup K_{M-1}}$, with $M - 1 \geq N$, has already been defined in such a way that $\sup_{K_j} |\tilde{x}| \leq \max_{1 \leq i \leq N} C^{i-j} \sup_{K_i} |x|$ whenever $N < j < M$. Define now $\tilde{x}$ on K_M. If $t \in D(M,j)$ for some $j < M$, set $x(t) = C^{j-M} x(\phi_{Mj}(t))$. Note that x is well-defined on $\cup_{j<M} D(M,j)$: if $t \in D(M,j) \cap D(M,i)$, then $C^{j-M} x(\phi_{Mj}(t)) = C^{i-M} x(\phi_{Mi}(t))$. Also, for such t, $|x(t)| \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} |x|$.
Moreover, $\tilde{x}$ is continuous on the closed set $D(M,j)$ for every $j < M$, and thus also on $\cup_{j<M} D(M,j)$. Extend $\tilde{x}$ to a continuous function on K_M without increasing the sup-norm.
Finally, set $\tilde{x}(\infty) = 0$. The function $\tilde{x}$ thusly defined belongs to X. Indeed, it is continuous on each of the sets K_n, and also at ∞, given that $\sup_{K_j} |\tilde{x}| \leq \mathrm{const} \cdot C^{-j}$. Finally, if $t \in D(n,m)$, then $\tilde{x}(t) = C^{m-n} \tilde{x}(\phi_{nm}(t))$.

(2) Modify the recursive process from part (1). Suppose $\tilde{x}|_{K_1 \cup \ldots \cup K_{M-1}}$, where $M - 1 \geq N$, has already been defined in such a way that $0 \leq \tilde{x} \leq y|_{K_1 \cup \ldots \cup K_{M-1}}$ on $K_1 \cup \ldots \cup K_{M-1}$ and $\sup_{K_j} \tilde{x} \leq \max_{1 \leq i \leq N} C^{i-j} \sup_{K_i} x$ whenever $N < j < M$. Define now $\tilde{x}$ on K_M. If $t \in D(M,j)$ for some $j < M$, set $x(t) = C^{j-M} x(\phi_{Mj}(t))$. As before, observe that x is well-defined on $\cup_{j<M} D(M,j)$. Clearly, for $t \in D(M,j)$,

$$0 \leq \tilde{x}(t) \leq y(t), \text{ and } \tilde{x}(t) \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} x.$$

Also, $\tilde{x}|_{\cup_{j<M} D(M,j)}$ is continuous. Therefore, we can find $u \in C(K_M)$ so that

$$\sup_{K_M} |u| = \sup_{\cup_{j<M} D(M,j)} |\tilde{x}| \leq \max_{1 \leq i \leq N} C^{i-M} \sup_{K_i} |x|.$$

To define $\tilde{x}$ on K_M, set $\tilde{x} = u \wedge y$. □

We shall use the notation $K'_n = K_n \backslash (\cup_{m<n} D(n,m))$, and $K' = \cup_n K'_n$ (note that these sets are open).
In a manner similar to the preceding lemma, one can prove:

Lemma 2.8. *Suppose $m \leq n$, $t \in K'_m$, $s \in K'_n$, and $U \subset K'_m$, $V \in K'_n$ are disjoint open sets with the property that $t \in U \subset \overline{U} \subset K'_m$ and $s \in V \subset \overline{V} \subset K'_n$. Then for $\alpha, \beta \in [0, \infty)$, there exists $x \in X_+$ so that:*

(1) For $j < m$, $x|_{K_j} = 0$.
(2) $x(t) = \alpha$, $x(s) = \beta$, $x \leq \alpha$ on U, and $x \leq \beta$ on V.
(3) If $m < n$, then $x|_{K_m \backslash U} = 0$.
(4) If $m < n$, then for $m < j < n$, $0 \leq x|_{K_j} \leq C^{m-j} \alpha$.
(5) On K_n, $0 \leq x \leq C^{m-n} \alpha \vee \beta$.
(6) For $j > n$, $0 \leq x|_{K_j} \leq (C^{m-j} \alpha) \vee (C^{n-j} \beta)$.

Proof. We shall consider the case of $m < n$ (that of $m = n$ is handled similarly). In light of Lemma 2.7, it suffices to construct a consistent family of functions $x_j \in C(K_j)$, with $j \leq n$, satisfying the properties listed above. For $j < m$, simply set $x_j = 0$. Define $x_m \in C(K_m)_+$ which vanishes outside of U and satisfies $0 \leq x \leq \alpha = x(t)$.

Use Lemma 2.7 to find $x_j \in C(K_j)$ so that the family $(x_j)_{j<n}$ is consistent and $x_j \leq C^{m-j}\alpha$.

Define $x_n \in C(K_n)$ in such a way that:

(1) $x_n = 0$ on ∂V, and $0 \leq x_n \leq \beta = x_n(s)$ on V.
(2) $x_n(t) = C^{j-n} x_j(\phi_{nj}(t))$ whenever $t \in D(n, j)$ for some $j < n$.

Such a function x_n exists, since $\overline{V}$ is disjoint from $\cup_{j<n} D(n, j)$. Furthermore, the family $(x_j)_{j\leq n}$ is consistent. To define x_j for $j > n$, again invoke Lemma 2.7. □

Next we consider "upward" extensions.

Lemma 2.9. *(1) If $L \leq N$, and $x \in C(K_L \cup \ldots \cup K_N)$ is a consistent function, then there exists a consistent $\tilde{x} \in C(K_1 \cup \ldots \cup K_N)$ so that $\tilde{x}|_{K_L \cup \ldots \cup K_N} = x$, and for $j < L$, $\sup_{K_j} |\tilde{x}| \leq \max_{L\leq i\leq N} C^{i-j} \sup_{K_i} |x|$.*

(2) If, furthermore, $y \in X_+$ is such that $0 \leq x \leq y$ on $K_L \cup \ldots \cup K_N$, then $\tilde{x}$ can be selected in such a way that, in addition, $0 \leq \tilde{x} \leq y$.

Proof. We only prove (1), as (2) is handled similarly (compare with the proof of Lemma 2.7).

Define $\tilde{x}$ recursively. Suppose $\tilde{x}|_{K_{M+1} \cup \ldots \cup K_N}$ $(M + 1 \leq L)$ has already been defined in such a way that $\sup_{K_j} |\tilde{x}| \leq \max_{L\leq i\leq N} C^{i-j} \sup_{K_i} |x|$ whenever $M < j < N$. Now define $\tilde{x}$ on K_M. If $t \in D(M, j)$ for some $j \in \{M + 1, \ldots, N\}$, set $x(t) = C^{j-M} x(\phi_{Mj}(t))$. Note that x is well-defined on $\cup_{N\leq j<M} D(M, j)$: if $t \in D(M, j) \cap D(M, i)$, then $C^{j-M} x(\phi_{Mj}(t)) = C^{i-M} x(\phi_{Mi}(t))$. Also, for such t, $|x(t)| \leq \max_{1\leq i\leq N} C^{i-M} \sup_{K_i} |x|$.

As $\tilde{x}|_{\cup_{M<j\leq N} D(M,j)}$ defined above is continuous, we can extend it to the whole K_M, without increasing the sup-norm. □

2.3. **Atoms in a Benyamini space.**

Definition 2.10. *A point $k \in K'$ is called* hereditarily isolated *if it is an isolated point of K'_n for some $n \in \mathbb{N}$, and $\phi_{nm}(k)$ is isolated in K_m whenever $k \in D(n, m)$.*

For a point k like this, we can define a function $\theta_k \in X$ by setting $\theta_k(k) = 1$, $\theta_k(\phi_{nm}(k)) = C^{n-m}$ whenever $k \in D(n, m)$, and $\theta_k(t) = 0$ otherwise. Clearly θ_k is a normalized atom in X. Our next result claims that all atoms in X are of this form.

Proposition 2.11. *If $x \in X$ is a normalized atom, then $x = \theta_k$ for some hereditarily isolated point k.*

Proof. Suppose $x \in X$ is a normalized atom. Find $k \in K_n'$ such that $x(k) = 1$. We now prove that k is a hereditarily isolated point and that $x = \theta_k$. In particular, we must show that if $k \in D(n,m)$, then $\phi_{nm}(x)$ is isolated in K_m (note that here, $m \geq n$ necessarily).

Suppose, for the sake of contradiction, that $k_m = \phi_{nm}(k)$ is not isolated in K_m for some m. Find the smallest such m. Find distinct $a_1, a_2 \in K_m$ so that $x(a_1), x(a_2) > 1/2$. Find $y \in C(K_m)$ so that $0 \leq y \leq x|_{K_m}$, $y_1(a_1) = \frac{1}{2}$, and $y(a_2) = 0$. By Proposition 2.5, there exists $\tilde{y} \in [0, x] \subset X$ such that $\tilde{y}|_{K_m} = y$. By our choice of y, $\tilde{y}$ cannot be a scalar multiple of x. Thus x is not an atom, which is the desired contradiction. □

2.4. **The dual of a Benyamini space.**

Lemma 2.12. *Let X and K' be as above. Then X^* is lattice isometric to $M(K')$.*

Proof. Any measure on K' determines a linear functional on X; this gives rise to a contraction $\mathbf{i} : M(K') \to X^*$. We prove that $\mathbf{i}$ is a surjective isometry by showing that any $x^* \in X^*$ can be represented by $\mu \in M(K')$ with $\|\mu\| \leq \|x^*\|$. By the Hahn-Banach Theorem, x^* extends to a functional on $C(K)$ of the same norm; the latter is implemented by a measure $\mu \in M(K)$, with $\|\mu\| = \|x^*\|$. By removing a point mass at ∞, we can and do assume that μ lives on $\cup_n K_n$.

We claim that μ vanishes on $K \backslash K'$. Indeed, otherwise find the smallest value of n for which μ does not vanish on $K_n \backslash K_n'$; then $\mu|_{\cup_{j<n} D(n,j)} \neq 0$. Find the smallest j so that $\mu|_{D(n,j)} \neq 0$. Then the measure

$$\mu' = \mu - \mu\big|_{D(n,j)} + C^{j-n} \mu\big|_{D(n,j)} \circ \phi_{jn}$$

implements the same functional x^*; here, for $x \in C(K)$, we define $\big[\mu\big|_{D(n,j)} \circ \phi_{jn}\big](x)$ to be $\mu\big|_{D(n,j)}\big(x\big|_{D(j,n)} \circ \phi_{nj}\big)$. Note that $\mu'(E) = \mu(E) + C^{j-n}\mu(\phi_{jn}(E))$ for $E \subset D(j,n)$, $\mu'(E) = 0$ for $E \subset D(n,j)$, and $\mu'(E) = \mu(E)$ if E is disjoint from $D(n,j) \cup D(j,n)$. Furthermore, $\mu'|_{K_m} = \mu|_{K_m}$ for $m \notin \{j, n\}$, $\mu'|_{K_n} = \mu|_{K_n \backslash D(n,j)}$, and $\mu'|_{K_j} = \mu|_{K_j} + C^{j-n}\mu\big|_{D(n,j)} \circ \phi_{jn}$. It follows that

$$\|\mu'|_{K_n}\| = \|\mu|_{K_n}\| - \|\mu|_{D(n,j)}\|,$$

while

$$\|\mu'|_{K_j}\| \leq \|\mu|_{K_j}\| + C^{j-n}\|\mu|_{D(n,j)}\|,$$

Therefore,

$$\|\mu'\| = \sum_i \|\mu'|_{K_i}\| = \|\mu'|_{K_n}\| + \|\mu'|_{K_j}\| + \sum_{i\notin\{j,n\}} \|\mu'|_{K_i}\|$$
$$\leq (C^{j-n}-1)\|\mu|_{D(n,j)}\| + \sum_i \|\mu|_{K_i}\| < \sum_i \|\mu|_{K_i}\| = \|x^*\|,$$

a contradiction.

It is clear that the map $\mathbf{i}$ is positive (a positive measure generates a positive functional). We now show that $\mathbf{i}$ is bipositive: if $\mu \in M(K')$ is not a positive measure, then the corresponding functional is not positive either. We can write $\mu = (\mu_n)$, with (μ_n) concentrated on K'_n. Note that $\|\mu\| = \sum_n \|\mu_n\|$. Find $N \in \mathbb{N}$ so that $\mu_n \geq 0$ for $n < N$, but μ_N is not positive. By the regularity of the measure μ_N, we can find a positive $x_N \in C(K_N)$, vanishing on $\cup_{j<N} D(N,j)$, so that $\mu_N(x_N) < 0$. By scaling, we can and do assume that $\|x_N\|_\infty = 1$. Let $\delta = -\mu_N(x_N)/3$. Find $M > N$ so that $\sum_{j>M} C^{N-j}\|\mu_j\| < \delta$.

For $j < N$, let x_j be the zero function on K_j. For $N < j \leq M$, find an open set $U_j \subset K_j$ containing $\cup_{i<j} D(j,i)$ with $\|\mu_j|_{U_j}\| < \delta/M$. Now use Lemma 2.7 to define, recursively, a consistent family of functions x_j $(j > N)$ so that $\|x_j\| \leq C^{N-j}$ and x_j vanishes outside of U_j for $N < j \leq M$. By our choice of U_j, we have $|\mu_j(x_j)| \leq \delta C^{N-j}/M$ for $N < j \leq M$; for $j > M$, we have $|\mu_j(x_j)| \leq \delta C^{N-j}\|\mu_j\|$. Merge all the x_j's into a function $x \in X$. Then

$$\mu(x) \leq \mu_N(x_N) + \sum_{j>N} |\mu_j(x_j)| \leq -3\delta + \sum_{j=N+1}^{M} C^{N-j}\frac{\delta}{M} + \sum_{j>M} C^{N-j}\|\mu_j\|$$
$$< -3\delta + (M-N+1)\frac{\delta}{M} + \sum_{j>M} C^{N-j}\|\mu_j\| < -3\delta + \delta + \delta = -\delta,$$

which shows that the linear functional determined by μ is not positive.

We have established that $\mathbf{i} : M(K') \to X$ is a bipositive surjective isometry. By [1], $\mathbf{i}$ is a lattice isometry. □

We shall denote by $\mathcal{A}_1$ the set of normalized atoms of X^*. By Lemma 2.12, $X^* = M(K')$, hence $\mathcal{A}_1 = \{\delta_t : t \in K'\} \subset \mathbf{B}(X^*)_+$. Below we show that $\mathcal{A}_1$ (equipped with the weak* topology inherited from X^*) is topologically homeomorphic to K'.

Lemma 2.13. *The map* $\mathbf{j} : K' \to \mathcal{A}_1 : t \mapsto \delta_t$ *is a homeomorphism.*

Proof. To establish the continuity of $\mathbf{j}$, suppose the net t_α converges to t in K'. By continuity, $\delta_{t_\alpha}(x) = x(t_\alpha) \to x(t) = \delta_t(x)$ for any $x \in X$, hence $\delta_{t_\alpha} \to \delta_t$ in the weak* topology.

For the continuity of $\mathbf{j}^{-1}$, consider a net $(t_\alpha) \subset \mathcal{A}_1$ so that $\delta_{t_\alpha} \to \delta_t \in \mathcal{A}_1$ in the weak* topology – that is, $x(t_\alpha) \to x(t)$ for any $x \in X$. By the compactness of K, it suffices to show that the limit of any convergent subnet of (t_α) is t.

Suppose (t'_β) is a subnet of (t_α), which converges to $s \in K$. Then for any $x \in X$, we have $x(s) = \lim_\beta x(t'_\beta) = x(t)$. As $x(t)$ is not always 0, part (2) of Definition 2.1 implies $s \neq \infty$. Further, $x(t) = x(s)$ for any $x \in X$, hence parts (3) and (4) of Definition 2.1 show that $t = s$.

□

3. Renormings of Benyamini spaces

Theorem 3.1. *Suppose $(X, \|\cdot\|)$ is a Benyamini space. Then, for any $c > 1$, X can be equipped with an equivalent norm $|||\cdot|||$ so that $\|\cdot\| \leq |||\cdot||| \leq c^2\|\cdot\|$, so that the identity is the only lattice isometry on $(X, |||\cdot|||)$. If X has no more than one atom, then $|||\cdot|||$ can be selected to be an AM-norm.*

Remark 3.2. The restriction on the number of atoms is essential here; see Remark 3.8.

The rest of this section is devoted to proving Theorem 3.1.

Assume that X is a C-Benyamini space ($C < 2$) and that $c < \sqrt[3]{C}$. Let A and B be the sets of all $n \in \mathbb{N}$ for which K'_n is infinite, resp. finite and non-empty. For $n \in B$, write $K'_n = \{t_{1n}, \ldots, t_{p_n n}\}$. For $n \in A$, find a sequence $t_{1n}, t_{2n}, \ldots$ of distinct elements of K'_n which is dense in K'_n. Find a family $(\lambda_{in})_{n \in A \cup B} \subset (1, c)$ of distinct numbers so that: (i) for $n \in A$, $c > \lambda_{1n} > \lambda_{2n} > \ldots$, and $\lim_i \lambda_{in} = 1$; (ii) for $n \in B$, $c > \lambda_{1n} > \ldots > \lambda_{p_n n} > 1$. For each $t \in K'$, let $\mu(t) = \lambda_{in}$ if $t = t_{in}$ for some i and n, $\mu(t) = 1$ otherwise.

Denote the normalized atoms of X by $(\theta_i)_{i \in I}$, where the set I is countable. By Proposition 2.11, each θ_i corresponds with a hereditarily isolated point $a_i \in K'$. Furthermore, for each i, there exists a canonical band projection P_i onto $\mathrm{span}[\theta_i]$. Then $P_i x = x(a_i)\theta_i$.

Our definition of $|||\cdot|||$ would depend on the cardinality of I.

$\underline{|I| = 0.}$ For $x \in X$ set

$$|||x||| = \sup_{t \in K'} \mu(t)|x(t)|. \tag{3.1}$$

$\underline{|I| = 1.}$ Write $I = \{1\}$; represent X as $X_1 \oplus \mathbb{R}$, where $X_1 = \ker P_1$ is a C-Benyamini space (with the underlying space obtained by removing from K all the points $\phi_{nm}(a_1)$, when $a_1 \in K_n$ and $m \geq n$). Let $|||\cdot|||_1$ be the norm defined on X_1 using (3.1) (with some collection (t_{ni})). Let

$$|||x||| = \max\big\{|||(I - P_1)x|||_1, \|P_1 x\|\big\}. \tag{3.2}$$

$\underline{|I| > 1.}$ Write $I = \{1, \ldots, m\}$ ($2 \leq m < \infty$) or $I = \mathbb{N}$. Let $\mathcal{P} = \{(i,j) \in I^2 : i < j\}$, and let $\pi : \mathcal{P} \to \mathbb{N}$ be an injection. For $(i,j) \in \mathcal{P}$, let $\|\cdot\|_{i,j}$ be the norm on $\mathbb{R}^2$ whose unit ball is an octagon with vertices

$$\left(\pm\left(1 - \frac{c-1}{c(2\pi(i,j)+1)}\right), \pm 1 \right) \text{ and } \left(\pm 1, \pm\left(1 - \frac{c-1}{2c\pi(i,j)}\right) \right)$$

We mention some properties of the norms $\|\cdot\|_{i,j}$, to be used in the sequel.

N1 $\|\cdot\|_\infty \leq \|\cdot\|_{i,j} \leq c\|\cdot\|_\infty$.

N2 The formal identity $(\mathbb{R}^2, \|\cdot\|_{i_1,j_1}) \to (\mathbb{R}^2, \|\cdot\|_{i_2,j_2})$ (with the first vector of the canonical basis mapping to the first, and the second – to the second) is an isometry iff $i_1 = i_2$ and $j_1 = j_2$. This follows from a comparison of extreme points.

N3 For $\gamma > 1$ and $k \in I$, there exists $L = L(k, \gamma) \geq k$ so that $\|\cdot\|_{k,j} \leq \gamma\|\cdot\|_\infty$ for $j > L$.

N4 For $\gamma > 1$, there exists $M = M(\gamma)$ so that $\|\cdot\|_{i,j} \leq \gamma\|\cdot\|_\infty$ whenever $j > i > M$.

N5 If $|\alpha| \vee |\beta| = 1$ and $|\alpha| \wedge |\beta| \leq 1/c$, then $\|(\alpha, \beta)\|_{ij} = 1$.

We let

$$\|\!|x|\!\| = \max\Big\{ \sup_{t \in K'} \mu(t)|x(t)|, \sup_{(i,j)\in\mathcal{P}} \big\| \big(\mu(a_i)x(a_i), \mu(a_j)x(a_j)\big) \big\|_{i,j} \Big\}. \tag{3.3}$$

Clearly, we always have $\|\cdot\| \leq \|\!|\cdot|\!\| \leq c^2\|\cdot\|$ (in fact, if $|I| \leq 1$, we can replace c^2 by c). It is also clear that for $|I| \leq 1$, $\|\!|\cdot|\!\|$ is an AM-norm. To show that the only lattice isometry on $(X, \|\!|\cdot|\!\|)$ is the trivial one, we need a series of lemmas. As the proof for $|I| = 1$ follows immediately from that for $|I| = 0$, we shall only consider the cases of $I = \emptyset$ and $|I| \geq 2$.

First we establish the norms of point masses. Let $\hat{\delta}_t = \mu(t)\delta_t$.

Lemma 3.3. *For any $t \in K'$, $\big\|\!\big|\hat{\delta}_t\big|\!\big\| = 1$.*

Proof. For $x \in X$ and $t \in K'$, we clearly have $\|\!|x|\!\| \geq \mu(t)|x(t)| = \big|\hat{\delta}_t(x)\big|$, hence $\big\|\!\big|\hat{\delta}_t\big|\!\big\| \leq 1$. It remains to prove the opposite inequality.

Fix $t \in K'$ and $\gamma > 1$. We need to find $x \in X_+$ such that $x(t) = 1/\mu(t)$ and $\|\!|x|\!\| \leq \gamma$. To this end, find n so that $t \in K_n'$. Next, construct a finite set $V \subset K_n'$ consisting of "potentially troublemaking" points. If $|I| = \emptyset$, let

$$V = \{s \in K_n' : \mu(s) > \gamma\mu(t)\}.$$

If $|I| \geq 2$ and t is not hereditarily isolated, let

$$V = \{s \in K_n' : \mu(s) > \gamma\mu(t)\} \cup \{a_i \in K_n' : i \leq M(\gamma)\},$$

with $M(\gamma)$ as in [N4].

If $|I| \geq 2$ and t is hereditarily isolated, then $t = a_k$ for some k. Let

$$V = \{s \in K_n' : \mu(s) > \gamma\mu(t)\} \cup \{a_i \in K_n' : i \leq M(\gamma) \vee L(k,\gamma)\} \backslash \{a_k\},$$

where $L(k, \gamma)$ comes from property [N3].

The set V is finite and does not contain t. Find an open set $U \subset K'_n \backslash V$ containing t. Find $x \in C(K_n)$ such that x vanishes outside of U and $0 \leq x \leq 1/\mu(t) = x(t)$. Define x to be 0 on K_m for $m < n$. This function is consistent, so by Proposition 2.5, there exists $\tilde{x} \in X_+$ so that $\tilde{x}|_{K_1 \cup \ldots \cup K_n} = x$ and $\|\tilde{x}\| = 1/\lambda_{in}$.

It remains to show that $|||\tilde{x}||| \leq \gamma^2$. This will follow if we establish that

$$\mu(s)|\tilde{x}(s)| \leq \gamma \text{ for any } s \in K', \tag{3.4}$$

and (in the case of $|I| \geq 2$)

$$\left\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\right\|_{i,j} \leq \gamma^2 \text{ for any } i < j. \tag{3.5}$$

Note that, due to our construction of $\tilde{x}$, $\tilde{x}(s) = 0$ if $s \in K'_m$ with $m < n$. For $s \in K'_n$, we have $\tilde{x}(s) = 0$ for $s \notin U$, while for $s \in U$, $\mu(s) \leq \gamma\mu(t)$, so $\mu(s)|\tilde{x}(s)| \leq \gamma$. Finally, if $s \in K'_m$ for some $m > n$, we have $\tilde{x}(s) \leq C^{n-m}/\mu(t)$, hence $\mu(s)|\tilde{x}(s)| \leq c/C < 1 < \gamma$. This establishes (3.4).

To handle (3.5), note that if $a_i \in \cup_{m<n} K'_m \cup (K'_n \backslash U)$, then $\tilde{x}(a_i) = 0$, and therefore,

$$\left\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\right\|_{i,j} = \left\|\big(0, \mu(a_j)\tilde{x}(a_j)\big)\right\|_{i,j} = \mu(a_j)\tilde{x}(a_j).$$

The right hand side cannot exceed γ, as discussed in the paragraph relating to (3.4). The same conclusion holds if $a_j \in \cup_{m<n} K'_m \cup (K'_n \backslash U)$.

If $a_i, a_j \in \cup_{\ell>n} K'_\ell$, then $\tilde{x}(a_i), \tilde{x}(a_j) \leq 1/(\mu(t)C)$, hence

$$\left\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\right\|_{i,j} \leq \frac{c^2}{\mu(t)C} < 1.$$

Now consider the case of $a_i \in U$, $a_j \in \cup_{\ell>n} K'_\ell$. In this situation, $\mu(a_j)\tilde{x}(a_j) < c/C < c^{-2}$, hence, by [N5],

$$\left\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\right\|_{i,j} \leq \gamma.$$

The same conclusion holds if $a_j \in U$, $a_i \in \cup_{\ell>n} K'_\ell$.

Finally, if $a_i, a_j \in U$, then $\mu(a_i), \mu(a_j) \leq \gamma\mu(t)$. By the choice of U,

$$\left\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\right\|_{i,j} \leq \gamma\left\|\big(\mu(a_i)\tilde{x}(a_i), \mu(a_j)\tilde{x}(a_j)\big)\right\|_\infty \leq \gamma^2.$$

The same conclusion holds if the roles of a_i and a_j are reversed. We have now established (3.5). □

Now suppose T is a surjective lattice isometry on $(X, |||\cdot|||)$. Note first that T fixes the atoms of X:

Lemma 3.4. *For any $i \in I$, $T\theta_i = \theta_i$.*

Proof. This is obvious if $|I| \leq 1$. For $|I| \geq 2$, let $e_i = \theta_i/\mu(a_i)$ be the normalized atoms. By (3.3), for any $\alpha, \beta \in \mathbb{R}$, we have

$$|||\alpha e_i + \beta e_j||| = \left\|(\alpha, \beta)\right\|_{i,j}.$$

If T maps e_i and e_j to e_k and e_ℓ respectively, then

$$\|(\alpha,\beta)\|_{i,j} = \|(\alpha,\beta)\|_{k,\ell} \text{ for any } \alpha,\beta,$$

which, in light of Property [N2], implies $i = k$, $j = \ell$. ☐

Now observe that T^* is interval preserving [12, Theorem 1.4.19], hence it maps atoms of X^* to atoms. The atoms in X^* are characterized by Proposition 2.11. By Lemma 3.3, the set of normalized atoms of $(X^*, |||\cdot|||)$ (which we shall denote by $\mathcal{A}$) coincides with $\{\hat{\delta}_t : t \in K'\}$.

Thus, by Lemma 3.3, there exists a bijection $\psi : K' \to K'$ so that $T^*\hat{\delta}_t = \hat{\delta}_{\psi(t)}$. We shall show that $\psi(t) = t$ is the identity map. In fact, Lemma 3.4 already shows that $\psi(t) = t$ if t is a hereditarily isolated point.

To proceed further, in the next few lemmas we examine weak* convergence in $\mathcal{A}$. For convenience, we denote by ϕ_{nn} the identity map on $D(n,n) := K_n$.

Lemma 3.5. *Suppose $m, n \in \mathbb{N}$, $t \in K'_n$, and the sequence $(t_i) \subset K'_m \backslash \{t\}$ converges to s. Then the following are equivalent:*

(1) $m \geq n$, and $s = \phi_{nm}(t)$.
(2) $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \alpha \hat{\delta}_t$ for some $\alpha > 0$.

Moreover, if (1) holds, then (2) holds with $\alpha = C^{n-m}/\mu(t)$.

Proof. To show that (1) implies (2), as well as the "moreover" statement, we only need to observe that, due to our selection of (λ_{jm}), we have $\lim_i \mu(t_i) = 1$. We need to establish the converse.

First show that $m \geq n$. If $m < n$, then find an open set $U \subset K'_n$ containing t. By Proposition 2.5, there exists $x \in X$ so that $0 \leq x \leq 1 = x(t)$, which vanishes on $K_n \backslash U$ and on K_j for $j < n$. In particular, $\hat{\delta}_t(x) \neq 0$, while $\hat{\delta}_{t_i}(x) = 0$ for any i. This contradicts (2).

Thus $m \geq n$. Next show that $t \in D(n,m)$ and $s = \phi_{nm}(t)$. Suppose, for the sake of contradiction, that either $t \notin D(n,m)$, or $t \in D(n,m)$ and $s \neq \phi_{nm}(t)$. Find the smallest $i \leq m$ so that $s \in D(m,i)$, and let $s' = \phi_{mi}(s)$. Then $t \neq s'$. By Lemma 2.8, there exists $x \in X$ so that $x(t) = 1$ and $x(s') = 0$, hence also $x(s) = 0$. We observe that $\hat{\delta}_t(x) \neq 0$ and $\lim_i \hat{\delta}_{t_i}(x) = 0$, again contradicting (2). ☐

Lemma 3.6. *Suppose we are given $t \in K'_n$ and a sequence $(t_i) \subset K' \backslash \{t\}$. Then the following are equivalent:*

(1) There exists $m \geq n$ so that for i large enough, $t_i \in K'_m$. Furthermore, (t_i) converges to $s = \phi_{nm}(t)$.
(2) $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \alpha \hat{\delta}_t$ for some $\alpha > 0$.

Moreover, if (1) holds, then, in (2), $\alpha = C^{n-m}/\mu(t)$.

Proof. Lemma 3.5 shows that (1) implies (2), as well as the "moreover" conclusion. To establish (2) $\Rightarrow$ (1), find, for each i, $m(i) \in \mathbb{N}$ so that $t_i \in K'_{m(i)}$. We shall show that the sequence $(m(i))$ is eventually constant.

First we show that $(m(i))$ is bounded. Indeed, otherwise we can find a sequence (i_p) so that $\lim_p m(i_p) = \infty$. Clearly $\lim x(t_{i_p}) = 0$ for any $x \in X$, hence $\hat{\delta}_{i_p} \stackrel{w^*}{\to} 0$.

Now suppose, for the sake of contradiction, that $(m(i))$ does not stabilize. Passing to a subsequence, we can assume that there exists $m_1 \neq m_2$ so that $m(i) = m_1$ if i is odd, and $m(i) = m_2$ is even if i is even. Further, we can assume that (t_{2i-1}) and (t_{2i}) converge to $s_1 \in K_{m_1}$ and $s_2 \in K_{m_2}$, respectively. From Lemma 3.5, $m_1, m_2 \geq n$, $t_{2i} \to s_2 = \phi_{m_2 n}(t)$, and $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \hat{\delta}_t/(C^{m_2-n}\mu(t))$. Similarly, $t_{2i-1} \to s_1 = \phi_{m_1 n}(t)$, and $\mathrm{w}^* - \lim_i \hat{\delta}_{t_i} = \hat{\delta}_t/(C^{m_1-n}\mu(t))$. Thus, $1/\alpha = C^{m_2-n}\mu(t) = C^{m_1-n}\mu(t)$, which leads to the impossible conclusion $m_1 = m_2$.

Thus, the sequence $(m(i))$ is eventually constant. To conclude the proof, invoke Lemma 3.5. $\square$

Lemma 3.7. *Suppose $t \in K'$ is not hereditarily isolated. Then there exists a sequence $(t_i) \subset K'$ so that $\hat{\delta}_{t_i} \stackrel{w^*}{\to} \alpha\hat{\delta}_t$, for some $\alpha \in (0, 1]$. Moreover, for every such sequence there exists $r \in \{0, 1, 2, \ldots\}$ so that $\alpha = 1/(C^r \mu(t))$.*

Proof. Suppose first t is not isolated in K_n. Then t cannot be isolated in the open subset $K'_n \subset K$, so we can find a sequence $(t_i) \subset K'_n$, converging to t. Clearly $\delta_{t_i} \to \delta_t$ (in the weak* topology). Moreover, $\mu(t_i) \to 1$, hence $\hat{\delta}_{t_i} \to \alpha\hat{\delta}_t$, where $\alpha = 1/\mu(t) \in (1/c, 1]$.

Now suppose t is isolated in K_n (equivalently, in K'_n). Use Proposition 2.11 to find the smallest $m > n$ s.t. $s = \phi_{nm}(t)$ is not isolated in K_m. We claim that K'_m is non-empty, and s belongs to the closure. Indeed, as $t \in K'_n$, s cannot belong to $D(m, k)$ with $k < n$. In addition, if $s \in D(m, k)$ for some $n \leq k < m$, then s is an isolated point of $D(m, k)$, due to the minimality of m. Consequently, s is an isolated point of $\cup_{k<m} D(m, k)$. As s is not isolated in K_m, we can find a sequence $(t_i) \subset K'_m$ converging to t. Then $\delta_{t_i} \stackrel{w^*}{\to} C^{n-m}\delta_t$, hence $\hat{\delta}_{t_i} \to \alpha\hat{\delta}_t$, where $\alpha = C^{n-m}/\mu(t) \in (C^{n-m}/c, C^{n-m}]$.

Now suppose $\hat{\delta}_{t_i} \stackrel{w^*}{\to} \alpha\hat{\delta}_t$, for some $\alpha \in (0, 1]$. By Lemma 3.6, there exists m so that $t_i \in K_m$, for m large enough; and furthermore, $t_i \to \phi_{mn}(t)$. As in the previous paragraph, $\alpha = C^{n-m}/\mu(t)$. $\square$

Theorem 3.1 – completion of the proof. Suppose T is a lattice isometry on $(X, |||\cdot|||)$. By Subsection 2.4, it suffices to show that $T^*\hat{\delta}_t = \hat{\delta}_t$ for any $t \in K'$. As T^* maps normalized atoms to normalized atoms, $T^*\hat{\delta}_t = \hat{\delta}_s$, where $s = \psi(t) \in K'$. By Lemma 3.4, $\psi(t) = t$ if t is hereditarily isolated. As the set $\mathcal{A}$ of normalized atoms is identified with $\left\{\hat{\delta}_t : t \in K'\right\}$, we conclude that t is not hereditarily isolated iff $\psi(t)$ satisfies the same condition. For future use, note that if t is hereditarily isolated, then $t = t_{in}$ for some i, n.

Now suppose t is not hereditarily isolated. Let $s = \psi(t)$. In light of Lemma 3.7, there exists a sequence $(u_i) \subset K'$ so that $\hat{\delta}_{u_i} \stackrel{w^*}{\to} \alpha\hat{\delta}_t$. Moreover, for every such sequence,

$$\frac{1}{\mu(t)} = \nu(t) := \sup\left\{C^k\alpha : k \in \{0, 1, 2, \ldots\}, C^k\alpha \le 1\right\}.$$

Being isometric and weak* to weak* continuous, T^* preserves $\nu(\cdot)$, hence $\mu(\psi(t)) = \mu(t)$, for any $t \in K'$.

Recall that t_{in} is the unique point t with $\mu(t) = \lambda_{in}$. Consequently, $\psi(t_{in}) = t_{in}$, or equivalently, $T^*\hat{\delta}_{t_{in}} = \hat{\delta}_{t_{in}}$.

Now suppose $t \in K' \backslash (\cup_{i,n}\{t_{in}\})$ is not hereditarily isolated. Find a sequence $\left(t_{i_j n_j}\right)_j$ which converges to $\phi_{mn}(t)$ for some $m \ge n$. By Lemma 3.5,

$$\mathrm{w}^* - \lim_j \hat{\delta}_{t_{i_j n_j}} = C^{n-m}\hat{\delta}_t,$$

hence, due to the weak* to weak* continuity of T^*,

$$\mathrm{w}^* - \lim_j T^*\hat{\delta}_{t_{i_j n_j}} = C^{n-m}\hat{\delta}_{\psi(t)},$$

However, the left hand sides of the two centered expressions coincide, hence $\psi(t) = t$. □

Remark 3.8. In Theorem 3.1, the desired renorming cannot be an AM-space if the number of atoms exceeds 1. Indeed, suppose $a_1, \ldots, a_n$ are normalized atoms in an AM-space X, and let $X_0 = \{a_1, \ldots, a_n\}^\perp$. If π is a permutation of $\{1, \ldots, n\}$, then $T : X \to X$, defined by $Ta_i = a_{\pi(i)}$ and $Tx = x$ for $x \in X_0$, is an isometry. Thus, any AM renorming of a space with more than one atom will have non-trivial lattice isometries.

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